INTERCOUPLED THERMOELASTICITY WITH A FINITE VELOCITY OF HEAT PROPAGATION

I. M. Shter

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General theorems are derived concerning intercoupled thermoelasticity with a finite velocity of heat propagation. The variational principle is applied and the solution to intercoupled problems is given in an integral form analogous to the Kirchhoff formula.

1. When the principles of thermodynamics of irreversible processes are applied to deformations of solids, one obtains fundamental laws which govern real processes in elastic bodies. In problems concerning the deformations of a medium which interacts with external fields, the external forces are not given as functions of the space coordinates of a point and of time but are established, instead, from the solution to systems of simultaneous equations: equations of the mechanics of deformable bodies and equations of the external fields. In such a formulation it becomes necessary to consider equations of both elasticity theory and heat conduction. The deformation of a body by mechanical or thermal forces is accompanied by a coupling effect due to interaction between the deformation field and the temperature field. This effect is manifested in the generation of thermoelastic waves, in the thermoelastic dissipation of energy.

The velocity of heat propagation, according to the formula $v_T = \sqrt{a/\tau_*}$, is in metals of the same order of magnitude as the velocity of sound and in polymers, dielectrics, and amorphous materials equal to the latter. For this reason, in problems of intercoupled dynamic thermoelasticity, where the velocity of sound is taken into consideration, it is necessary to consider also the velocity of heat propagation. Assuming an infinite velocity of heat propagation in dynamic problems of thermoelasticity results, formally, in the appearance of stresses at a given point before the elastic wave has arrived; this becomes very obvious in the analysis of a thermal shock at a half-space surface.

The equations of intercoupled thermoelasticity with a finite velocity of heat propagation are

$$\nabla^2\Theta = \frac{1}{a}\cdot\frac{\partial\Theta}{\partial t} + \frac{1}{v_T^2}\cdot\frac{\partial^2\Theta}{\partial t^2} + \eta\left(\frac{\partial e}{\partial t} + \tau^*\frac{\partial^2 e}{\partial t^2}\right), \tag{1}$$

$$\sigma_{iK,K} + F_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2}$$

where Θ denotes the temperature rise, $\eta = T_0\alpha_T(\lambda + 2\mu/3)$, T_0 denotes the temperature before heating, σ_{ik} is the stress tensor, F_i are volume forces, and ρ denotes the density of the medium. To this system must be added the equation of coupling between the deformation tensor and the components of the displacement vector:

$$e_{iK} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_K} + \frac{\partial u_K}{\partial x_i}\right). \tag{3}$$

We will derive the energy equation for intercoupled thermoelasticity with heat sources in the medium. Let us consider such a system of equations:

$$\sigma_{iK,K} + F_i = \rho\dot{v}_i, \quad v_i = \dot{u}_i = \frac{\partial u_i}{\partial t} \quad (i,\ K = 1,\ 2,\ 3), \tag{4}$$

$$\Theta_{,KK} - \frac{1}{a}\dot{\Theta} - \frac{1}{v_T^2}\ddot{\Theta} - \eta(\dot{e} + \tau^*\ddot{e}) = -\frac{1}{a}\left(1 + \tau^*\frac{\partial}{\partial t}\right)Q, \tag{5}$$

where Q(x, t) denotes the intensity of heat sources.

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We multiply (4) by v_i, integrate over the volume according to the Gauss theorem, and obtain

$$\int_B F_i v_i dV + \int_A P_i v_i ds = \rho \int_B \dot{v}_i v_i dV + \int_B \sigma_{iK} \dot{e}_{iK} dV, \tag{6}$$

where B denotes the body volume, A denotes the surface bounding the body, and P_i denotes the surface loading.

The Duhamel–Neumann formula

$$\sigma_{iK} = 2\mu e_{iK} + (\lambda e - \gamma\Theta)\,\delta_{iK}, \tag{7}$$

transforms (6) into

$$\frac{d}{dt}(K + W) = \int_B F_i v_i dV + \int_A P_i v_i ds + \gamma \int_B \Theta \dot{e} dV. \tag{8}$$

Here

$$K = \frac{\rho}{2}\int_B v_i v_i dV, \quad W = \int_B \left(\mu e_{iK} e_{iK} + \frac{\lambda}{2} e^2\right) dV, \quad \gamma = \alpha_T (3\lambda + 2\mu). \tag{9}$$

Equation (8) expresses the law of energy conservation in a thermoelastic medium, but it does not explicitly account for the presence of heat sources and for the temperature rise in the body. Transforming (5) and introducing the thermal energy function $P = \gamma/2\eta a \int_B \Theta^2 dV$ as well as the dissipation function

$$\chi_T = \varkappa T_0 \int_B \left(\frac{\Theta_{,i}}{T_0}\right)^2 dV + a\tau^* \int_B \Theta \ddot{s} dV, \tag{10}$$

we obtain the energy equation

$$\frac{d}{dt}(K + W + P) + \chi_T = \int_B F_i v_i dV + \int_A P_i v_i ds + \frac{C_e}{T_0}\int_B \Theta (Q + \tau^* \dot{Q})\, dV + \frac{\varkappa}{T_0}\int_A \Theta\Theta_{,n} ds \tag{11}$$

(s denotes the entropy). The right-hand side of this equation includes heat sources which produce a deformation field and a temperature field. Our dissipative function differs from Biot's analogous dissipative function by the additional term representing the accelerated increase of the system entropy. When assuming an infinite velocity of heat propagation, therefore, we have less dissipated energy than in the case of a finite velocity.

With the aid of Eq. (11), one can prove the uniqueness of the solution to intercoupled thermoelasticity problems with constraints and with a finite velocity of heat propagation.

2. The application of direct methods to the solution of intercoupled thermoelasticity problems is fraught with difficulties; on the other hand, approximate methods based on variational principles are effective.

We will establish the variational principle for intercoupled thermoelasticity. Considering the isothermal deformation energy

$$W = \int_B \left(\mu e_{iK} e_{iK} + \frac{\lambda}{2} e^2\right) dV \tag{12}$$

then transforming this expression by means of the Duhamel–Neumann formula and the equation of motion, we obtain the relation

$$\int_B F_i \delta u_i dV + \int_A P_i \delta u_i ds - \rho \int_B \ddot{u}_i \delta u_i dV = \delta W - \gamma \int_B \Theta \delta e dV. \tag{13}$$

The left-hand side of Eq. (13) represents the virtual work of volume, surface, and inertia forces, while the right-hand side represents the virtual work of internal forces.

Noting that the second term on the right-hand side of Eq. (13) includes the temperature, we add another equation here. Vector **H** will be related to the thermal flux vector **q** and the entropy as follows:

$$\mathbf{q} = T_0\dot{\mathbf{H}}, \quad s = -\operatorname{div}\mathbf{H}, \quad \dot{q} = T_0\ddot{\mathbf{H}}, \tag{14}$$

and then the integral

$$\int_B \left(\Theta_{,i} + \frac{T_0}{\varkappa}\dot{\mathbf{H}}_i + \frac{T_0\tau^*}{\varkappa}\ddot{\mathbf{H}}_i\right)\delta\mathbf{H}_i dV = 0 \tag{15}$$

can be reduced to

$$\int_A \Theta_{n_i}\delta\mathbf{H}_i ds + \frac{c_e}{T_0}\int_B \Theta\delta\Theta dV + \gamma\int_B \Theta\delta e dV + \frac{T_0}{\varkappa}\int_B (\dot{\mathbf{H}}_i + \tau^*\ddot{\mathbf{H}}_i)\, dV = 0. \tag{16}$$

We also introduce the thermal potential P and the dissipation functions D. Considering (16) and (13), we have the variational principle stated as follows:

$$\delta(W + P + D) = \int_B (F_i - \rho\ddot{u}_i)\,\delta u_i dV + \int_A P_i\delta u_i ds - \int_A \Theta_{n_i}\delta\mathbf{H}_i ds, \tag{17}$$

saying that the variation of the sum of the deformation work, the thermal potential, and the dissipation function is equal to the virtual work of external forces, the virtual work of inertia forces, and the surface heating. The thermal potential and the dissipation function have been defined according to

$$\delta P = \frac{c_e}{T_0}\int_B \Theta\delta\Theta dV, \quad \delta D = \frac{T_0}{\varkappa}\int_B (\dot{\mathbf{H}}_i + \tau^*\ddot{\mathbf{H}}_i)\,\delta\mathbf{H}_i dV. \tag{18}$$

The components of the displacement vector $\mathbf{u}_i$ and the components of vector H_i will be represented as follows:

$$\mathbf{u}_i = \sum_{j=1}^{n} u_{ij}(x_K)\, q_j(t), \quad \mathbf{H}_i = \sum_{j=1}^{n} H_{ij}(x_K)\, q_j(t), \tag{19}$$

where q_j are generalized coordinates, δu_i and δH_i will be assumed independent of time, so that the following respective definitions will apply:

$$\delta u_i = \frac{\partial u_i}{\partial q_j}\delta q_j, \quad \delta H_i = \frac{\partial H_i}{\partial q_j}\delta q_j, \quad \frac{\partial H_i}{\partial q_j} = \frac{\partial \dot{H}_i}{\partial \dot{q}_j},$$

$$\delta K = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{q}_j}\right]\delta q_j, \quad \delta u_i = v_i dt, \quad \delta\Theta = \dot{\Theta} dt.$$

Principle (17) can be expressed in terms of Lagrange equations of motion for energy dissipating systems

$$\frac{\partial W}{\partial q_j} + \frac{\partial D_T}{\partial \dot{q}_j} + \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j} + \tau^*\frac{\partial D_T}{\partial \dot{q}_j}\right) = Q_j, \tag{20}$$

where $D_T = T_0/2\varkappa \int_B (\dot{H}_i)^2 dV$ and Q_j is the generalized force:

$$Q_j = \int_B F_i\frac{\partial u_i}{\partial q_j} dV + \int_A \left(P_i\frac{\partial u_i}{\partial q_j} - \Theta_{n_i}\frac{\partial \dot{H}_i}{\partial \dot{q}_j}\right) ds. \tag{21}$$

If the intensity of the entropy source is defined as

$$\sigma = -\frac{1}{T_0}\int_B q\Theta_{,i} dV$$

and the surface intensity of the entropy source is introduced as

$$\sigma_n = -\frac{1}{T_0}\int_A \Theta_{n_i} q ds, \quad \sigma_n > 0, \tag{22}$$

then (17) can be expressed as

$$\frac{d}{dt}(K+W+P)+\sigma-\sigma_n=\int\limits_B F_i v_i dV+\int\limits_A P_i v_i ds. \tag{23}$$

Equation (23) can be integrated as follows: without surface heating, the work of volume and surface forces is expended on changing the kinetic energy, the isothermal deformation energy, the thermal function, and also on increasing the system entropy (at $\sigma_n = 0$); with surface heating, $\sigma_n \neq 0$ and the system entropy decreases, i.e., less energy is dissipated under a mechanical load on the body.

3. We will now represent the displacements as sums of a potential and a solenoidal component:

$$u_i=\Phi_{,i}+\epsilon_{ijK}\psi_{K,j}, \tag{24}$$

so that the system of intercoupled thermoelasticity equations in dimensionless variables

$$\tau=\frac{c_1^2}{a}t,\qquad \varepsilon_i=\frac{c_1}{a}x_i \tag{25}$$

becomes

$$\left(\nabla^2-\frac{\partial^2}{\partial\tau^2}\right)\Phi(\varepsilon_i,\ \tau)=m\Theta(\varepsilon_i,\ \tau), \tag{26}$$

$$\left(\nabla^2-k^2\frac{\partial^2}{\partial\tau^2}\right)\psi_i(\varepsilon_i,\ \tau)=0, \tag{27}$$

$$\left(\nabla^2-\frac{\partial}{\partial\tau}-M^2\frac{\partial^2}{\partial\tau^2}\right)\Theta(\varepsilon_i,\ \tau)-\eta_1\left(\frac{\partial}{\partial\tau}+M^2\frac{\partial^2}{\partial\tau^2}\right)\nabla^2\Phi=0, \tag{28}$$

where

$$c_1^2=\frac{\lambda+2\mu}{\rho};\quad c_2^2=\frac{\mu}{\rho};\quad m=\frac{\alpha_T a^2(3\lambda+2\mu)}{c_1^4\rho};$$

$$\eta_1=\frac{T_0\alpha_T c_1^2(3\lambda+2\mu)}{c_e a};\quad k^2=\left(\frac{c_1}{c_2}\right)^2;\quad M=\frac{c_1}{v_T}.$$

A Laplace transformation of (26) and (28) with respect to the variable τ, with homogeneous initial conditions, will yield

$$(\nabla^2-p^2)\overline{\Phi}=m\overline{\Theta}, \tag{29}$$

$$[\nabla^2-p(1+M^2p)]\overline{\Theta}-\eta_1 p(1+M^2p)\nabla^2\overline{\Phi}=0. \tag{30}$$

Eliminating function $\overline{\Phi}$ from (29) and (30) yields

$$\{(\nabla^2-p^2)[\nabla^2-p(1+M^2p)]-\varepsilon p(1+M^2p)\nabla^2\}\overline{\Theta}=0, \tag{31}$$

where $\varepsilon = m\eta_1$ is the coupling parameter.

Let us consider the solution to the equation

$$\{(\nabla^2-p^2)[\nabla^2-p(1+M^2p)]-\varepsilon p(1+M^2p)\nabla^2\}\times\overline{H}(\varepsilon_i,\ y,\ p)=-(1+M^2p)\delta(\varepsilon_i-y) \tag{32}$$

in an infinite region. With homogeneous initial conditions, the solution to (32) is

$$\overline{H}=\frac{e^{-\lambda_1 r}-e^{-\lambda_2 r}}{4\Pi r(\lambda_1^2-\lambda_2^2)(1+M^2p)}, \tag{33}$$

where

$$\lambda_{1,2}^2=\frac{p}{2}\left[p+(1+\varepsilon)(1+M^2p)\pm\sqrt{p^2-2p(1-\varepsilon)(1+pM^2)+(1+\varepsilon)^2(1+pM^2)^2}\right],$$
$$r^2\equiv(\varepsilon_i-y_i)(\varepsilon_i-y_i). \tag{34}$$

We write down the following identity:

$$\int\limits_B[\overline{H}(\nabla^2-\lambda_1^2)(\nabla^2-\lambda_2^2)\overline{\Theta}-\overline{\Theta}(\nabla^2-\lambda_1^2)(\nabla^2-\lambda_2^2)\overline{H}]\,dV$$

$$= \int_B [\bar{H}\nabla^4\bar{\Theta} - \bar{\Theta}\nabla^4\bar{H} - (\lambda_1^2 + \lambda_2^2)(\bar{H}\nabla^2\bar{\Theta} - \bar{\Theta}\nabla^2\bar{H}]\, dV. \tag{35}$$

With the Gauss theorem and the Green transformation, we then obtain

$$\int_B [\bar{H}(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\bar{\Theta} - \bar{\Theta}(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\bar{H}]\, dV$$

$$= \int_A \left(\bar{H}\frac{\partial\Box\bar{\Theta}}{\partial n} - \bar{\Theta}\frac{\partial\Box\bar{H}}{\partial n}\right) ds - \int_A \left[\nabla^2\bar{\Theta}\frac{\partial\bar{H}}{\partial n} - \nabla^2\bar{H}\frac{\partial\bar{\Theta}}{\partial n}\right] ds, \tag{36}$$

where $\Box^2 = \nabla^2 - (\lambda_1^2 + \lambda_2^2)$. From (36) we have

$$\bar{\Theta}(y, p) = \frac{1}{1 + M^2 p}\int_A \left(\bar{H}\frac{\partial\Box^2\bar{\Theta}}{\partial n} - \bar{\Theta}\frac{\partial\Box^2\bar{H}}{\partial n}\right) ds$$

$$+ p\eta_1 \int_A \left(\nabla^2\bar{G}\frac{\partial\bar{\Theta}}{\partial n} - \nabla^2\bar{\Phi}\frac{\partial\bar{H}}{\partial n}\right) ds, \tag{37}$$

where $\bar{G}$ is the transform of the Green function which corresponds to the thermoelastic potential $\bar{\Phi}$,

$$\nabla^2\bar{\Phi} = \frac{[\nabla^2 - p(1 + M^2 p)]\bar{\Theta}}{\eta_1 p(1 + M^2 p)}, \quad \Box^2 = \nabla^2 - [\lambda_1^2 + \lambda_2^2 - p(1 + M^2 p)]$$

or

$$\bar{\Theta}(y, p) = \frac{1}{1 + M^2 p}\left[\int_A \left(\bar{H}\frac{\partial\Box\bar{\Theta}}{\partial n} - \bar{\Theta}\frac{\partial\Box\bar{H}}{\partial n}\right) ds\right.$$

$$\left. - \int_A \left(\nabla^2\bar{\Theta}\frac{\partial\bar{H}}{\partial n} - \nabla^2\bar{H}\frac{\partial\bar{\Theta}}{\partial n}\, ds\right]\right..$$

Function $H(\varepsilon_i, y, \tau)$ will be sought in the following form:

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots \tag{38}$$

Expanding (34) into a Maclaurin series in ε and retaining only the terms of not higher than the first order in ε, we have

$$\lambda_1 = p\left[1 + \varepsilon\frac{1 + pM^2}{2(p - 1 - pM^2)}\right],$$

$$\lambda_2 = \sqrt{p(1 + pM^2)}\left(1 - \varepsilon\frac{pM^2 + 1}{2(p - 1 - pM^2)}\right). \tag{39}$$

Inserting (39) into (33) and a subsequent Laplace transformation yield

$$H_0 = -H(\tau - r)\left[1 + M^2 e^{-\frac{\tau - r}{M^2}} + e^{\frac{\tau - r}{1 - M^2}}\right]$$

$$- \int_0^\tau \left(e^{\frac{\tau - \delta}{1 - M^2}} - e^{-\frac{\tau - \delta}{M^2}}\right)\left[e^{-\frac{r}{2M}} + \frac{r}{2M}\int_{rM}^{\delta} e^{-\frac{r\zeta}{2M}} \frac{J_1\left(\frac{1}{2M^2}\sqrt{\zeta^2 - r^2M^2}\right)}{\sqrt{\zeta^2 - r^2M^2}}\, d\zeta\right] d\delta \tag{40}$$

at $\tau > rM$ or

$$H_0 = -H(\tau - r)\left[1 + M^2 e^{-\frac{\tau - r}{M^2}} + e^{\frac{\tau - r}{1 - M^2}}\right] \tag{41}$$

at $\tau < rM$.

The expressions for H_1 and H_2 have an analogous structure, but are not shown here because of their unwieldiness. Having determined the values of H and knowing the temperature of the body surface, one can calculate the temperature inside the body according to Eq. (37). This equation is analogous to the well known Kirchhoff equation of elastokinetics. Such an equation is also obtained in the case of mechanical loads on the body.

Equation (40) indicates that, when the body boundary heats up, a thermoelastic wave propagates as follows: a thin layer of material first heats up, then expands, and then becomes a source of an elastic compression wave (41), whereupon a thermal wave travels through time $\tau > rM$ and raises the temperature of the medium at a given point.

We obtain a thermoelastic wave, subject to damping and dispersion, which is not characteristic for solutions of nonintercoupled problems of thermoelasticity.

At M = 1, according to (40) and (41), we obtain discontinuous solutions, i.e., when the velocity of heat propagation is equal to the velocity of sound, then shock waves are generated by external impulses and, therefore, additional conditions must be stipulated, if it is to become feasible to apply the equations of continuum mechanics.

The appearance of shock waves is characteristic when a finite velocity of wave propagation is assumed, while at an infinite heat velocity only a discontinuity of stresses occurs during a continuous change of temperature, a so-called isothermal shock, which has certainly no physical justification.

NOTATION

Θ is the temperature;
v_T is the velocity of thermal wave;
a is the thermal diffusivity;
λ, μ are the Lamé constants;
u_i are displacements;
Q is the intensity of heat sources;
q is the thermal flux;
$\varkappa$ is the thermal conductivity;
τ^* is the relaxation time
c_e is the specific heat at a constant deformation;
α_T is the linear thermal expansivity;
e is the first invariant of the deformation tensor.

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